MULTIPLICITIES OF THE MOST SINGULAR POINT ON SCHUBERT VARIETIES ON GL(N)/B FOR $n = 5, 6$

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ABSTRACT. We calculate using Macaulay 2 the multiplicities of the most singular point on Schubert varieties on Gl(n)/B for $n = 5, 6$. The method of computation is described and tables of the results are included.

1. INTRODUCTION

In this paper, we compute, using Macaulay 2, the multiplicity of the most singular point on Schubert varieties of the flag manifold $Gl(n)/B$ for $n = 5$ and $n = 6$, using a description of the preimage of Schubert varieties in $GL(n)$ first given by Fulton [3] and more recently developed by Knutson and Miller [5]. Results of Krattenthaler [6], Rosenthal and Zelevinsky [9], and Lakshmibai and Weyman [8] give combinatorial and determinantal formulas for multiplicities (at all points) of Schubert varieties on the Grassmanian. Furthermore, the singular loci of Schubert varieties on flag manifolds have been much studied, with known results collected in [1], starting with the fundamental result of Lakshmibai and Sandhya [7] that a Schubert variety indexed by the permutation w is singular iff w contains either the pattern 1324 or the pattern 2143. However, no results for the multiplicities of the Schubert varieties appear to be known on the full flag variety.

In the next section, we briefly define the objects under study and outline some basic results about them. A detailed introduction to Schubert varieties can be found, for example, in [4, part III]. This section also serves to fix our conventions; the differing choices made by different authors in the subject can cause significant confusion. In particular, our convention for indexing Schubert varieties is opposite to the convention used in [1], which now seems to be fairly standard. The third section describes our algorithm, and the fourth section demonstrates this algorithm for $w = 1324$. Two appendices give our code and the results of our computations.

2. DEFINITIONS AND CONVENTIONS

A *(complete) flag* $\mathcal{F}$ in $\mathbb{C}^n$ is a sequence of subspaces $\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$ such that the subspace F_i has dimension i. Fixing a basis for $\mathbb{C}^n$, we can represent $\mathcal{F}$ by a matrix M as follows. For each component F_i of $\mathcal{F}$, pick a vector $m_i \in F_i \setminus F_{i-1}$, and write m_i as the i-th row of M. Note that F_i will be the span of the first i rows of M; in particular, M is in invertible since $F_n = \mathbb{C}^n$.

This representation of the flag is clearly not unique, as it involves repeated choices of vectors. To be precise, both multiplying any row of M by a constant and adding any row to a subsequent row of M leaves the flag unchanged. This is equivalent to multiplying M on the left by a lower triangular matrix. We can now give the structure of an algebraic variety to the set of flags; namely, it is the quotient of $G = Gl(n)$ by the group of lower triangular matrices B^- acting on the left, and we

denote this variety by $B^-\backslash G$ and call it the *flag variety*. We also have a natural map

$$\pi : G \to B^-\backslash G$$

sending a matrix to the flag it represents.

We can, however, pick a standard representation $N(\mathcal{F})$ for each flag $\mathcal{F}$ as follows. Pick a matrix M that represents $\mathcal{F}$. Now take the leftmost nonzero entry in the first row of $M = [m_{ij}]$, which we call m_{1l}, and use elements of B^- to change M so that $m_{kl} = 0$ for $k > 1$. We then take the leftmost nonzero entry of the second row, place 0s in all entries below it, and repeat for all rows in order to finally get $N(\mathcal{F})$. Unfortunately, we cannot identify $B^-\backslash G$ with these representations of flags except as a set, since this process of row reduction destroys the geometry of $B^-\backslash G$.

Let the group B^+ of upper triangular matrices act on G by right multiplication; by associativity of multiplication, this commutes with the left action of B^- and therefore gives an action on $B^-\backslash G$. This action can be thought of as adding any column of a matrix to a column to its right. Under these two actions, any matrix can be sent to a unique permutation matrix W, so we can index orbits of the right action of B^+ on $B^-\backslash G$ by permutations. In particular, a flag $\mathcal{F}$ is in the orbit of $\pi(W)$, where W is the permutation matrix with 1s in the leftmost nonzero entries of each row in $N(\mathcal{F})$. Let X_w denote the orbit of the flag $\pi(W^{-1})$, where W is the permutation w written as a matrix. X_w is known as a *Schubert cell*. The process of choosing the representative $N(\mathcal{F})$ for each flag does preserve geometry locally on X_w, so X_w is isomorphic to $\mathbb{A}^{\binom{n}{2}-l(w)}$, where $l(w)$ is the length of a shortest expression of w as a product of adjacent transpositions, or, equivalently, the number of inversions in w.

We denote by Y_w the closure of X_w in $B^-\backslash G$; it is known as a *Schubert variety*. For permutations $v, w \in S_n$, let $v \succ w$ if $l(v) > l(w)$ and $v = tw$ for some transposition t. The transitive closure of the relation $\succ$ is known as the *Bruhat order*; for the remainder of this paper, $v > w$ for $v, w \in S_n$ means that v is greater than w in this partial order. It is a classical result that

$$Y_w = \bigcup_{w' > w} X_{w'}.$$

Note that the unique 0-dimensional cell X_{w_0}, where $w_0 = n \cdots 21$ is contained in every Schubert variety.

Given a variety X, the *multiplicity* at a point p of X, which we will denote $\operatorname{mult}_p X$ is the degree of the projective tangent cone $\operatorname{Proj}(\operatorname{gr}_{\mathfrak{m}_p} \mathcal{O}_{X,p})$ as a subvariety of the projective tangent space $\operatorname{Proj}(\operatorname{Sym}^* \mathfrak{m}_p/\mathfrak{m}_p^2)$, or, equivalently, if the Hilbert–Samuels polynomial of $\mathcal{O}_{X,p}$ is written $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $\operatorname{mult}_p X = n! a_n$. For a nonsingular point p, $\operatorname{mult}_p X = 1$; at a singular point q, $\operatorname{mult}_q X > 1$ and measures roughly how singular X is at q. Slightly more precisely, the multiplicity counts how many times a generic hyperplane cuts through X in a neighborhood of q.

Since a Schubert variety is invariant under the right action of B^+, its multiplicity must remain constant on B^+ orbits, or Schubert cells. Moreover, since for $v \succ w$, there exists a $\mathbb{P}^1$ with one point in X_v and the remaining points in X_w, by semicontinuity, multiplicities must be nondecreasing with respect to Bruhat order. X_{w_0} must therefore be the most singular point of Y_w and the multiplicity of Y_w

there measures how singular Y_w gets. In particular, a multiplicity of 1 at X_{w_0} indicates that Y_w is smooth.

3. Explanation of the Algorithm

Since multiplicity is a local property, we can calculate it after restricting to an affine neighborhood of X_{w_0} in $B^-\backslash G$. A natural candidate is Ω_{w_0}, the orbit of W_0 (defined to be w_0 written as a permutation matrix and considered as a flag) under the right action of B^-. (In general, Ω_w is defined as the orbit of the flag $\pi(W^{-1})$ under the right action of B^- (rather than B^+) and is known as a *dual Schubert cell.*) Locally on Ω_{w_0}, the map $\pi : G \to B^-\backslash G$ has a section σ, namely the map that sends a flag $\mathcal{F}$ to $N(\mathcal{F})$. This identifies Ω_{w_0} with the matrices with 1s on the main antidiagonal and 0s to the right and below; X_{w_0} is mapped to the permutation matrix W_0. Since σ is a local section, Y_w is locally isomorphic in a neighborhood of X_{w_0} to $\pi^{-1}(Y_w) \cap \sigma(\Omega_{w_0})$.

Now we need to find equations defining $\pi^{-1}(Y_w)$. Fix a permutation $w \in S_n$. Let $R(w) = [r_{ij}(w)]$ be the integer matrix with $r_{ij}(w) = \#\{w^{-1}(k) \leq i, k \leq j\}$. For any matrix M, let M_{ij} denote the submatrix consisting of the first i rows and first j columns. Then, for any invertible matrix M with $\pi(M) \in X_w$, the rank of the submatrix M_{ij} will be r_{ij}. The proof of this claim is as follows. Note that the rank of M_{ij} for any i and j does not change under multiplication by B^- on the left, since the effect of multiplication by $b \in B^-$ on the first i rows is the same as that of multiplying by an element of $Gl(i)$, namely the submatrix of b consisting of the first i rows and columns. Therefore, the claim can be verified on matrices of the form $N(\mathcal{F})$ for $\mathcal{F} \in X_w$, where it is trivial.

It is a nontrivial combinatorial fact that Bruhat order can be equivalently defined by $v > w$ if $r_{ij}(v) \leq r_{ij}(w)$ for all i, j. Therefore $\pi^{-1}(Y_w)$ consists of all invertible matrices M satisfying the rank conditions $\mathrm{rk}(M_{ij}) \leq r_{ij}(w)$. Now let $Z = [z_{ij}]$ be a matrix of indeterminates, and I_w be the ideal generated by all size $1 + r_{ij}(w)$ minors of Z_{ij}, for all i and j. (As shown in [3], where I_w was originally defined, a small subset of the minors suffices to generate I_w, but we will not use this fact.) By the above statement, it is clear that I_w vanishes precisely on the points of $\pi^{-1}(Y_w)$; Knutson and Miller [5] prove that I_w is in fact radical, so $I_w = I(\pi^{-1}(Y_w))$. Now, by sending z_{ij} to 0 for entries below the main antidiagonal and 1 for entries on the main antidiagonal, we have the ideal J_w for $\pi^{-1}(Y_w) \cap \sigma(\Omega_{w_0})$ as a subvariety of $\sigma(\Omega_{w_0}) \cong \mathbb{A}^{\binom{n}{2}}$.

In our coordinates for $\sigma(\Omega_{w_0})$, X_{w_0} corresponds to the point where $z_{ij} = 0$ for all i and j, or equivalently the maximal graded ideal $\mathfrak{m} = \langle z_{ij} \rangle$. The projective tangent cone of Y_w is therefore $\mathrm{Proj}(\mathrm{gr}_{\mathfrak{m}}\, S/J_w)$, where $S = k[z_{ij}]$. For a polynomial $f \in S$, let f_d be the homogeneous part of degree d, and let $s(f)$ be the smallest number such that $f_{s(f)} \neq 0$. Let $J'_w = \langle f_{s(f)} | f \in J_w \rangle$; then $\mathrm{gr}_m\, S/J_w \cong S/J'_w$, since any polynomial f is sent to $f \pmod{\mathfrak{m}^{s+1}}$, of which f_s is a representative. Degree is invariant under Gröbner deformation and easiest to calculate on monomial ideals, so, to calculate the multiplicty of Y_w, or, equivalently, the degree of S/J'_w, it suffices to calculate the degree of $S/\operatorname{in}(J'_w)$ under any (graded) term order. This is the same as taking the initial ideal of J_w, taking care to use a term order which places the **lowest** degree term first.

In practice, on Macaulay 2, one does this by homogenizing the generators of J_w using a new variable t (or by sending z_{ij} to t rather than 1 for entries on the main

diagonal in passing from I_w to J_w), and using a term order that refines the partial order by degree in t. (See [2, Prop 15.28] for a proof that this is equivalent.) We can then compute the initial ideal, send t to 1, and calculate the degree.

4. Example

As an example, we calculate the multiplicity of Y_w at X_{w_0} for $w = 2143$, the smallest nontrivial example. Here, we have

$$W^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, we have the rank matrix

$$R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Exactly two of the rank entries give minors in I_w, namely $r_{11} = 0$, which gives $z_{11} \in I_w$, and $r_{33} = 2$, which gives

$$\det \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} = 0.$$

Setting $z_{33} = 0$ and $z_{32} = z_{23} = 1$, we have $J_w = \langle z_{11}, -z_{11} + z_{12}z_{31} + z_{21}z_{13} - z_{31}z_{22}z_{13}\rangle = \langle z_{11}, z_{12}z_{31} + z_{21}z_{13} - z_{31}z_{22}z_{13}\rangle$. Then $J'_w = \langle z_{11}, z_{12}z_{31} + z_{21}z_{13}\rangle$, and it is clear that the multiplicity of Y_w at X_{w_0} is 2. For purposes of illustration, we carry out the remainder of the algorithm. Homogenizing the generators of J_w gives us $\langle z_{11}, tz_{12}z_{31} + tz_{21}z_{13} - z_{31}z_{22}z_{13}\rangle$, and one possible appropriate initial ideal is $\langle z_{11}, tz_{21}z_{13}\rangle$. Sending t to 1, we get $\langle z_{11}, z_{21}z_{13}\rangle$, which has degree 2; therefore, we conclude that $\operatorname{mult}_{X_{w_0}} Y_w = 2$.

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Appendix A. Maple and Macaulay 2 code

The following is our Maple code for generating the rank matrices.

```
with(combinat);
with(linalg);
interface(prettyprint=false);
n:=5;

sum(matrix(n,n,
      (i,j)->`if`((j>=k) and (i>=op(k, pPerm)), 1, 0)), k=1..n);
```

The Maple output was sent to a file, and all square brackets [] were converted to curly brackets {} for Macaulay 2. Macaulay 2 code is as follows. Note that the first five lines must be changed for each n.

```
R=QQ[t,x11,x12,x21,x13,x22,x31,x14,x23,x32,x41,
       MonomialOrder=>Eliminate 1];

G = matrix({{x11,x12,x13,x14,t},
            {x21,x22,x23,t,0},
            {x31,x32,t,0,0},
            {x41,t,0,0,0},
            {t,0,0,0,0}})

S=QQ[x11,x12,x21,x13,x22,x31,x14,x23,x32,x41];

f=map(S,R,{1,x11,x12,x21,x13,x22,x31,x14,x23,x32,x41});

n=5;

# Mlist = (paste from Maple output)

# compute J_w
Ilist = apply(Mlist,
          M->trim(sum(flatten(for i from 0 to n-1 list
                               for j from 0 to n-1 list
                                 minors(M_(i,j)+1,
                                   submatrix(G, {0..i}, {0..j}))))));

GBlist = apply(Ilist, gb);

# gives in(J_w)
LTlist = apply(GBlist, GB -> leadTerm(gens(GB)));

# gives in(J^\prime_w)
ELTlist = apply(LTlist, LT->f(LT));

# outputs degrees
Dlist = apply(ELTlist, LT -> degree(ideal(LT))
```

The output of the last line gives the degrees. Output from other lines, such as the initial ideals, could also be of interest. These computations run quite quickly; in fact, generating the ideals J_w was by far the slowest step, and the Gröbner basis computation took only a few seconds for $n = 6$.

Appendix B. Computational Results for $n = 5$ and $n = 6$

We have listed the permutations by multiplicity of Y_w at X_{w_0}. For each multiplicity, permutations are listed in lexicographic order.

Multiplicity	Permutations
5	14325
3	13425, 14235, 24153, 31524
2	12435, 13245, 13254, 13524, 14253, 14352, 15324, 21354, 21435, 21453, 21534, 21543, 23154, 24135, 24315, 24351, 25143, 31254, 31425, 31542, 32154, 32514, 32541, 41325, 42153, 51324, 52143
1	12345, 12354, 12453, 12534, 12543, 13452, 13542, 14523, 14532, 15234, 15243, 15342, 15423, 15432, 21345, 23145, 23415, 23451, 23514, 23541, 24513, 24531, 25134, 25314, 25341, 25413, 25431, 31245, 31452, 32145, 32415, 32451, 34125, 34152, 34215, 34251, 34512, 34521, 35124, 35142, 35214, 35241, 35412, 35421, 41235, 41253, 41352, 41523, 41532, 42135, 42315, 42351, 42513, 42531, 43125, 43152, 43215, 43251, 43512, 43521, 45123, 45132, 45213, 45231, 45312, 45321, 51234, 51243, 51342, 51423, 51432, 52134, 52314, 52341, 52413, 52431, 53124, 53142, 53214, 53241, 53412, 53421, 54123, 54132, 54213, 54231, 54312, 54321

For $n = 6$, we have omitted the permutations of multiplicities 1 and 2 for reasons of space. (There are 366 nonsingular Schubert varieties and 207 with multiplicity 2 at X_{w_0}.) These can be inferred from the remainder with the help of the pattern avoidance criterion.

Multiplicity	Permutations
14	154326
10	153426
9	145326, 154236
8	321654
7	135426, 143526, 152436, 153246, 254163, 416325
6	145236, 132546, 214365
5	125436, 135246, 142536, 143256, 143265, 143625, 146325, 153264, 154263, 154362, 164325, 215436, 251364, 251436, 251463, 253164, 254136, 254316, 254361, 314625, 315426, 316425, 413625, 415326, 426153, 514326, 614325
4	153624, 152346, 134526, 214635, 215364, 215463, 216435, 231564, 231654, 241365, 243165, 245163, 312645, 312654, 314265, 321564, 321645, 326154, 351426, 351624, 413265, 416235, 421653
3	124356, 124365, 124536, 125346, 132564, 132645, 132654, 134256, 134265, 134625, 135264, 136425, 142356, 142365, 142635, 145263, 145362, 146235, 152364, 152463, 153462, 163425, 164235, 214356, 214536, 215346, 216453, 216543, 231546, 235164, 241536, 241563, 241635, 241653, 245136, 245316, 245361, 246153, 251346, 251634, 251643, 253146, 253416, 253461, 264153, 312546, 314526, 315246, 315264, 315624, 316245, 316254, 316524, 321546, 325164, 341625, 351264, 351642, 352164, 352614, 352641, 361524, 425163, 412635, 413526, 415236, 416253, 416352, 421635, 423165, 426135, 426315, 426351, 431625, 432165, 513426, 514236, 524163, 531624, 613425, 614235, 624153, 631524

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